Joe Dinius
Midterm

A ME 549
October 21, 2011

Problem 1 *(30%)* Indicate whether each of the statements below are *true* or *false*. Justify your answer by showing the validity of the statement or providing a counterexample, whichever is appropriate.

1. If a solution $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is nontrivial, then there exists $(T, J) \in \operatorname{dom}\phi$ such that $T + J > 0$.

   Solution.

   Claim. *This statement is true.*

   Proof. Assume the statement is not true. That is, $\nexists (T, J) \in \operatorname{dom}\phi$ s.t. $T + J > 0$. t, j where $(t, j) \in \operatorname{dom}\phi$ are initially 0 and cannot decrement. Therefore, (T, J) must be s.t. $T + J = 0 \implies T, J = 0$. So, $\operatorname{dom}\phi = \{(0, 0)\}$. Any solution of this form must be such that there is neither flow nor jump. This is a contradiction.

   Therefore, if a solution $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is nontrivial, then there exists $(T, J) \in \operatorname{dom}\phi$ such that $T + J > 0$. □

2. Every complete discrete solution is Zeno.

   Solution.

   Claim. *This statement is true.*

   Proof. Recall that a solution is Zeno if it is complete and $\sup_t \operatorname{dom}\phi < \infty$. The completeness is easily established, as the assumption is that the discrete solution is complete. Since the solution is discrete, $\sup_t \operatorname{dom}\phi = 0 < \infty$.

   Therefore, every complete discrete solution is Zeno. □

3. When $C \cap D = \emptyset$, there is no solution from points in ∂C.

   Solution.

   Claim. *This statement is false.*

   Proof. Take Example 2.8 from the notes, where $C = \mathbb{R}^2 \setminus D$, $D = \{x \in \mathbb{R}^2 : 0 \leq x_2 \leq -\frac{1}{5}x_1 + \frac{2}{5}\}$. Notice, $C \cap D = \emptyset$ by construction. Pick an initial condition that lies along the line $x_2 = -\frac{1}{5}x_1 + \frac{2}{5}$, which is part of the boundary ∂C. For simplicity, take $(x_1, x_2) = (2, 0)$ initially. Since $(2, 0) \in D$, the solution ϕ jumps to $(\frac{3}{4}, \frac{1}{4})$, which is in C. The solution ϕ then flows for all t thereafter.

   Therefore, by construction, a solution exists when the initial condition comes from ∂C and $C \cap D = \emptyset$. □

4. Even when f and g are single-valued and continuously differentiable functions, solutions to hybrid systems $\mathcal{H}$ may be non-unique.

   Solution.

   Claim. *This statement is true.*

   Proof. Take Example 2.8 from the notes again. f and g are continuously-differentiable, single-valued functions. Take the initial condition $(0, -1)$. Two solutions exist from this point, they are

$$\phi_1(t,0) = \begin{pmatrix} t \\ t-1 \end{pmatrix} \quad \phi_1(t,1) = \begin{pmatrix} (t-1) + \frac{3}{4} \\ (t-1) + \frac{1}{4} \end{pmatrix}$$

$$\phi_2(t,0) = \begin{pmatrix} t \\ t-1 \end{pmatrix} \quad \phi_2(t,j) = \begin{pmatrix} (\frac{3}{4})^j \\ (\frac{1}{4})^j \end{pmatrix}$$

   where $\operatorname{dom} \phi_1 = ([0,1] \times \{0\}) \cup ([1,\infty) \times \{1\})$ and $\operatorname{dom} \phi_2 = ([0,1] \times \{0\}) \cup (\{1\} \times \mathbb{N})$.

   Therefore, even when f and g are single-valued and continuously differentiable functions, solutions to hybrid systems $\mathcal{H}$ may be non-unique. □

Problem 2 *(30%)* Design the data (C, F, D, G) of a hybrid system that captures the dynamics of a mobile robot on the plane, with position vector $\xi \in \mathbb{R}^2$, moving around a circle N times in the clockwise direction and then stopping. The radius of the circle is to be determined by the norm of the initial position vector. A turn corresponds to crossing of the positive ξ_2 axis.

Solution. First, assume that the robot moves with constant angular velocity ω. Moving in the plane with constant angular velocity in the clockwise direction has the following state equation (the same as the simple harmonic oscillator)

$$f(\xi) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

I need to integrate a timer τ so that once the robot has completed N cycles, it will stop. At this time, I would like to perform the jump map, which will reset the timer and zero out the angular velocity. Therefore, I will have the state vector

$$x = \begin{pmatrix} \xi \\ \tau \\ \omega \end{pmatrix}.$$

So, once $\tau = 1$, I would like to jump via the map

$$g(x) = \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}.$$

That is, the robot will stop wherever it started.

Let's define the flow and jump sets first:

$$\begin{aligned} C &= \{(\xi, \tau, \omega) \in \mathbb{R}^2 \times [0,\ 1] \times \mathbb{R}_{\geq 0} : \xi_1^2 + \xi_2^2 = R^2\} \\ D &= \{(\xi, \tau, \omega) \in \mathbb{R}^2 \times \{1\} \times \mathbb{R}_{\geq 0} : \xi_1^2 + \xi_2^2 = R^2\}. \end{aligned}$$

Back to the counter, τ. I would like to scale the timer rate so that when $\tau = 1$, the jump map is applied. The scaling is simple enough

$$\dot{\tau} = \frac{2\pi N}{\omega}.$$

Therefore, the flow map will incorporate the SHO nature of the walking robot trajectory, the timer, and the constancy of ω during flow periods. That is,

$$F(x) = \begin{pmatrix} f(\Xi) \\ [0, 1] \\ 0 \end{pmatrix}$$

where Ξ is the set of all points at radius R in the plane (note that $f(\Xi) = \Xi$). The jump map is

$$G(x) = \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}.$$

□

Problem 3 *(40%)* A mechanical system consists of a point mass impacting with a vertical wall. Assuming unitary mass for sake of simplicity, the system is described by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - k_2 x_2 - f_c(x), \end{aligned}$$

where $x_1 \in \mathbb{R}$ denotes the horizontal position, $x_2 \in \mathbb{R}$ denotes the horizontal velocity, $k_1, k_2 \in \mathbb{R}$, f_c is the contact force given by

$$f_c(x) = \begin{cases} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

in which $k_c > 0$ and $b_c > 0$ are, respectively, the elastic and damping coefficients of the compliant contact model. When a collision with the surface located at $x_1 = 0$ occurs with a velocity of the mass greater or equal than $\bar{x}_2$, the impact is assumed to be impulsive and, accordingly, the rigid body instantaneously rebounds or jumps. The name value of the state variables after the impact is described by the reset law

$$\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= -\varrho x_2 \end{aligned}$$

where ϱ represents the restitution coefficient.

1. Derive a hybrid system model for the mechanical system.

   Solution. The system will jump whenever $x_1 = 0$ and $x_2 \geq \bar{x}_2$. So,

   $$D = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = 0,\ x_2 \geq \bar{x}_2\}.$$

   Define the flow set to be the complement of D,

   $$C = \mathbb{R}^2 \setminus D.$$

   The flow map is

   $$f(x) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix} & \text{if } x_1 \leq 0 \\ \begin{pmatrix} 0 & 1 \\ -(k_1 + k_c) & -(k_2 + b_c) \end{pmatrix} & \text{if } x_1 > 0 \end{cases}$$

   The jump map is

   $$g(x) = \begin{pmatrix} x_1 \\ -\varrho x_2 \end{pmatrix}.$$

   □

2. Consider the function $V(x) = \frac{1}{2}x^T x$ defined for each $x \in \mathbb{R}^2$ and the set $\mathcal{A} = \{(0,0)\}$. Determine the conditions of Theorem 3.18 in the notes that are satisfied and, if appropriate, the conditions on the parameters $k_1, k_2, k_c, b_c, \bar{x}_2, \varrho$.

   Solution. $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. In Theorem 3.18, property (3.2a) follows very easily, that is (for s the standard Euclidean norm and $x \in C \cup D \cup G(D)$, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ s.t.

   $$\alpha_1(s) \leq V(x) \leq \alpha_2(s).$$

   This property is easy, since $\alpha_1(s) = \alpha_2(s) = \frac{s^2}{2}$ foots the bill. Property (3.2b) is a little trickier (in fact, I don't believe it will hold).

   Taking the time-derivative,

   $$\begin{aligned} \langle \nabla V, f \rangle = \dot{V}(x) &= \dot{x}_1 x_1 + \dot{x}_2 x_2 \\ &= \begin{cases} (1 - k_1)x_1 x_2 - k_2 x_2^2 & \text{if } x_1 \leq 0 \\ (1 - (k_1 + k_c))x_1 x_2 - (k_2 + b_c)x_2^2 & \text{if } x_1 > 0 \end{cases} \end{aligned}$$

   The problem is this: the $x_1 x_2$ term, as there is no way to control the sign of this term with a constant such as k_1 or k_c. I could force the term to go to zero, with $k_1 = 1$, $k_c = 0$, but I think that I can do better (see the next part of this problem). I will claim that, with this choice of $V(x)$, property (3.2b) is not upheld.

For property (3.2c), check (recalling that $x_1 = 0$ for all points in D)

$$\begin{aligned} V(g(x)) - V(x) &= \begin{cases} -\frac{k_2}{2}(1-\varrho^2)x_2^2 & \text{if } x_1 \le 0 \\ -\frac{(k_2+b_c)}{2}(1-\varrho^2)x_2^2 & \text{if } x_1 > 0 \end{cases} \\ &= \begin{cases} -\frac{k_2}{2}(1-\varrho^2)(x_1^2+x_2^2) & \text{if } x_1 \le 0 \\ -\frac{(k_2+b_c)}{2}(1-\varrho^2)(x_1^2+x_2^2) & \text{if } x_1 > 0 \end{cases}. \end{aligned}$$

So, pick

$$\rho(s) = \begin{cases} \frac{k_2}{2}(1-\varrho^2)s^2 & \text{if } x_1 \le 0 \\ \frac{(k_2+b_c)}{2}(1-\varrho^2)s^2 & \text{if } x_1 > 0 \end{cases}$$

and property (3.2c) is upheld, provided $k_2 > 0$ and $k_2 > -b_c$. Also, assume the coefficient of restitution is dissipative ($\varrho < 1$). □

3. What can you say about the stability and attractively properties of $\mathcal{A} = \{(0,0)\}$.

Solution. I will try to get a better $V(x)$ starting with $V(x) = x^T P x$ for some yet to be determined positive definite and symmetric matrix P. I want to solve the matrix equation $A^T P + PA = -I$, where I is the 2×2 identity matrix. Solving for the terms in the matrix and assuming $p_{12} = p_{21}$, I get the following for P

$$P = \begin{cases} \begin{pmatrix} \frac{k_2^2+k_1(k_1+1)}{2k_1k_2} & \frac{1}{2k_1} \\ \frac{1}{2k_1} & \frac{k_1+1}{2k_1k_2} \end{pmatrix} & \text{if } x_1 \le 0 \\ \begin{pmatrix} \frac{(k_2+b_c)^2+(k_1+k_c)(k_1+k_c+1)}{2(k_1+k_c)(k_2+b_c)} & \frac{1}{2(k_1+k_c)} \\ \frac{1}{2(k_1+k_c)} & \frac{k_1+1}{2(k_1+k_c)(k_2+b_c)} \end{pmatrix} & \text{if } x_1 > 0 \end{cases}$$

To ensure that P is positive-definite, I require the following

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0. \end{aligned}$$

These conditions put the following restrictions on the system parameters (provided $k_1, k_2, (k_1 + k_c), (k_2 + b_c) \neq 0$)

$$\begin{aligned} k_2^2 + k_1(k_1+1) &> 0 \\ (k_2+b_c)^2 + (k_1+k_c)(k_1+k_c+1) &> 0. \end{aligned}$$

Now, $\dot{V}(x) = -x^T x$ (really makes the messy algebra worth it!), and property 3.2b is upheld. Even though not explicitly mentioned, property 3.2a is upheld using the eigenvalues of P, there are two $\lambda_1 < \lambda_2$. Take $\alpha_1(s) = \lambda_1 s^2$ and $\alpha_2(s) = \lambda_2 s^2$.

For property (3.2c),

$$\begin{aligned} V(g(x)) - V(x) &= (g(x))^T P g(x) - x^T P x \\ &= -p_{22}(1-\varrho^2)x_2^2 \\ &= -p_{22}(1-\varrho^2)(x_1^2+x_2^2) \text{ since } x_1 = 0 \text{ for } x \in D. \end{aligned}$$

Take $\rho(s) = \frac{k_1+1}{2k_1k_2}(1 - \varrho^2)s^2$ to satisfy property (3.2c). This places the added restriction that $k_1 > -1$, but $k_1 \neq 0$. As before, $\varrho < 1$.

Therefore, $\mathscr{A}$ is UGpAS. I can go a step further and claim that, since the domain of any solution to the hybrid system is unbounded, $\mathscr{A}$ is UGAS. $\square$